

# Hyperentropic systems and the generalized second law of thermodynamics

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(Dated: August 4, 2011)

The holographic bound asserts that the entropy  $S$  of a system is bounded from above by a quarter of the area  $\mathcal{A}$  of a circumscribing surface measured in Planck areas:  $S \leq \mathcal{A}/4\ell_P^2$ . This bound is widely regarded a desideratum of any fundamental theory. Moreover, it was argued that the holographic bound is necessary for the validity of the generalized second law (GSL) of thermodynamics. However, in this work we explicitly show that hyperentropic systems (those violating the holographic entropy bound) do exist in higher-dimensional spacetimes. We resolve this apparent violation of the GSL and derive an upper bound on the area of hyperentropic objects.

The past four decades have witnessed a breakthrough in computer and data storage technology [1]. The impressive reduction in size of information storage devices is one of the most remarkable advances in this field. It is believed that individual atoms and molecules may one day become short term information-storage devices [1]. Can this trend of miniaturization continue indefinitely? One (see [1, 2]) naturally wonders if there is some fundamental limitation on the size devices of given information capacity may reach in the future? As emphasized by Bekenstein [1, 2], such question is related to the following question: what are the limitations on the magnitude of the entropy of a system characterized by general parameters such as size and energy [2]?

The celebrated holographic principle of 't Hooft [3, 4] asserts that there is a deep connection between the physical content of a theory defined in a spacetime and the corresponding content of another theory defined on the boundary of the same spacetime [5]. A consistency requirement on the holographic principle is that the boundary of any physical system should be able to encode as much information as required to enumerate all possible quantum states of the bulk system [5]. In light of the correspondence between information and entropy [6], and the well-known entropy-area relation for black holes [7, 8], this requirement has been translated into the holographic entropy bound [3–5]. This bound asserts that the entropy  $S$  (or information) that can be contained in a physical system is bounded in terms of the area  $\mathcal{A}$  of a surface enclosing it [3–5]:

$$S \leq \frac{\mathcal{A}}{4\ell_P^2}, \quad (1)$$

where  $\ell_P^2 = G\hbar/c^3$  is the Planck area. [We shall henceforth use natural units in which  $G = c = k_B = 1$ .]

The holographic principle [3, 4] and the holographic bound (1) are widely regarded as guidelines to the ultimate physical theory of nature [9]. For systems in three spatial dimensions, the bound (1) suggests an information content that scales no faster than the area of the boundary of the space. The holographic bound also implies that  $(3+1)$ -dimensional black holes have the largest

possible entropy among (stationary and bounded) physical systems characterized by a given surface area  $\mathcal{A}$  (see also [10–12]).

As support for the holographic bound in *three* spatial dimensions, Susskind [4, 5, 13] described the following gedanken experiment: Take a neutral nonrotating spherical object of radius  $R$ , energy  $E$  (with  $R > 2E$ ), and entropy  $S$  which violates the holographic bound:  $S > \pi R^2/\hbar$ . A spherically symmetric and concentric shell of mass  $R/2 - E$  is dropped on the system; according to Birkhoff's theorem the total mass is now  $R/2$ . When the outermost surface of the shell reaches Schwarzschild radial coordinate  $r = R$ , the system becomes a black hole of radius  $R$  and entropy  $S_{\text{BH}} = \pi R^2/\hbar$ , which is *smaller* than the original entropy  $S$  [5]. Susskind argued that the apparent violation of the generalized second law of thermodynamics (GSL) [7] in this gedanken experiment should be regarded as evidence that the envisaged system cannot really exist.

Before we proceed, it should be mentioned that it is possible to find examples for systems which violate the holographic bound. For example, a collapsed object already inside its own gravitational radius eventually violates it. The enclosing area can only decrease while the enclosed entropy can only grow [5, 14, 15]. Another example is given by a large spherical section of a flat Friedmann universe: its enclosing area grows like radius squared while the enclosed entropy does so like radius cubed. These examples belong to a class of strongly self-gravitating and dynamical systems. The second example also describes an unisolated system. Nevertheless, it has been established [5] that the holographic bound (1) can be trusted for generic weakly self-gravitating isolated systems in three spatial dimensions. In the present work we shall focus on such weakly self-gravitating isolated systems.

Clearly, Susskind's gedanken experiment can also be applied to physical systems in higher-dimensional spacetimes. The arguments of [4] thus suggest that the holographic bound (1) must follow from the GSL in any number of spatial dimensions. But is the holographic bound really valid for physical systems in *higher*-dimensional spacetimes?

Proliferation of large spatial dimensions is expected to increase the entropy content of a physical system which is characterized by a given amount of energy. Evidently the more the dimensions, the more ways there are to split up a given amount of energy between the quantum states of the system [16]. Thus, one may expect the challenge to the holographic entropy bound to become more and more serious as the number of spatial dimensions increases. As an example, consider in  $D$  flat spatial dimensions a spherical box of radius  $R$  which contains massless fields. We shall follow the analysis of [16] in order to calculate the system's entropy in the thermodynamic regime.

The mean thermal energy in the sphere from one helicity degree of freedom is [16]

$$E_{\text{d.o.f}} = V_D(R) \int_0^\infty \frac{\hbar\omega}{(e^{\beta\hbar\omega} \mp 1)(2\pi)^D} d\omega, \quad (2)$$

where the upper (lower) signs correspond to boson (fermion) fields, and  $\beta \equiv 1/T$  is the inverse temperature of the system. Here

$$V_D(R) = \frac{2\pi^{D/2}}{D\Gamma(D/2)} R^D \quad (3)$$

is the volume of a sphere of radius  $R$  in  $D$  spatial dimensions, and

$$dV_D(\omega) = [2\pi^{D/2}/\Gamma(D/2)]\omega^{D-1}d\omega \quad (4)$$

is the volume in frequency space of the shell  $(\omega, \omega + d\omega)$ . We note that the distribution  $\omega^D/(e^{\beta\hbar\omega} \mp 1)$  in Eq. (2) peaks at the characteristic frequency

$$\bar{\omega} = \frac{D}{\hbar\beta} [1 \mp e^{-D} + O(e^{-2D})]. \quad (5)$$

From Eqs. (2)-(4) and the relation

$$\int_0^\infty \frac{x^D dx}{e^x \mp 1} = \zeta(D+1)\Gamma(D+1) \times \begin{cases} 1 & \text{for bosons;} \\ 1 - 2^{-D} & \text{for fermions,} \end{cases} \quad (6)$$

where  $\zeta(z)$  is the Riemann zeta function, one finds that the mean energy of all massless fields is given by

$$E = \frac{2N\zeta(D+1)\Gamma(\frac{D+1}{2})R^D}{\pi^{1/2}\Gamma(\frac{D}{2})\beta^{D+1}\hbar^D}, \quad (7)$$

where  $N$  is the number of massless degrees of freedom (the number of polarization states). Massless scalars contribute 1 to  $N$ , massless fermions contribute  $1 - 2^{-D}$  to  $N$  [16], an electromagnetic field contributes  $D - 1$  to  $N$  [17], and the graviton contributes  $(D+1)(D-2)/2$  to  $N$  [17]. Solving Eq. (7) for  $\beta\hbar/R$  one finds

$$\beta\hbar/R = C_D(N\hbar/RE)^{\frac{1}{D+1}}, \quad (8)$$

where

$$C_D \equiv \left[ \frac{2\zeta(D+1)\Gamma(\frac{D+1}{2})}{\pi^{1/2}\Gamma(\frac{D}{2})} \right]^{\frac{1}{D+1}}. \quad (9)$$

Likewise, one can write the thermal entropy of one helicity degree of freedom as [16]

$$S_{\text{d.o.f}} = V_D(R) \int_0^\infty \left[ \mp \ln(1 \mp e^{-\beta\hbar\omega}) + \frac{\beta\hbar\omega}{e^{\beta\hbar\omega} \mp 1} \right] \frac{dV_D(\omega)}{(2\pi)^D}. \quad (10)$$

After some algebra we obtain

$$S = \frac{2N(D+1)\zeta(D+1)\Gamma(\frac{D+1}{2})R^D}{\pi^{1/2}D\Gamma(\frac{D}{2})\beta^D\hbar^D} \quad (11)$$

for the total entropy of the system. Comparing (7) and (11), one deduces the relation

$$S = \frac{D+1}{D}\beta E. \quad (12)$$

Substituting Eq. (8) into Eq. (12), one finds

$$S = C_D(1 + 1/D)N^{\frac{1}{D+1}}(RE/\hbar)^{\frac{D}{D+1}} \quad (13)$$

for the  $(D+1)$ -dimensional radiation entropy.

It is important to emphasize that our analysis is appropriate only for weakly self-gravitating systems. In particular, formula (13) for the entropy can be trusted provided the system's energy (for a given radius  $R$ ) is bounded from above as here stated. The spacetime outside the spherical box (for  $D \geq 3$ ) is described by the  $(D+1)$ -dimensional Schwarzschild-Tangherlini metric [18, 19] of ADM energy  $E$ :

$$ds^2 = -H(r)dt^2 + H(r)^{-1}dr^2 + r^2d\Omega^{(D-1)}, \quad (14)$$

with

$$H(r) = 1 - \left(\frac{r_g}{r}\right)^{D-2}. \quad (15)$$

Here

$$r_g = \left[ \frac{16\pi E}{(D-1)A_{D-1}} \right]^{\frac{1}{D-2}} \quad (16)$$

is the gravitational radius of the system and

$$A_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)} \quad (17)$$

is the area of a unit  $(D-1)$ -sphere.

For the system to be weakly self-gravitating, one should impose the criterion  $H(r=R) \simeq 1$  at the surface of the sphere, or equivalently  $(r_g/R)^{D-2} \ll 1$ . Taking cognizance of Eqs. (15)-(16), this condition yields the restriction

$$RE \ll \frac{D-1}{16\pi}\mathcal{A}, \quad (18)$$

where  $\mathcal{A} = A_{D-1}R^{D-1}$  is the surface area of the system. We characterize this restriction by the dimensionless control parameter  $\eta$  defined by

$$\eta \equiv \frac{16\pi RE}{(D-1)\mathcal{A}} \ll 1. \quad (19)$$

Taking cognizance of Eqs. (13) and (19), we can write the system's thermal entropy as

$$S = C_D(1 + 1/D)N^{\frac{1}{D+1}} \left[ \frac{\eta(D-1)\mathcal{A}}{16\pi\hbar} \right]^{\frac{D}{D+1}}. \quad (20)$$

From Eq. (20) one learns that for the system's entropy to beat the holographic bound (that is,  $S > \mathcal{A}/4\hbar$ ), its surface area must be bounded from above according to

$$\frac{\mathcal{A}}{\hbar} < [4C_D(1 + 1/D)]^{D+1} N \left[ \frac{\eta(D-1)}{16\pi} \right]^D. \quad (21)$$

The validity of the thermodynamic description rests on the assumption that *many* quanta (of each degree of freedom) are thermally excited in the system:  $E/N \gg \hbar\omega$ . Taking cognizance of Eqs. (5) and (7), one finds the thermodynamic condition

$$C_D^{D+1}(R/\beta\hbar)^D \gg D. \quad (22)$$

Using the relation (8), one can cast this condition in the form

$$C_D^{-1}(N\hbar/RE)^{\frac{D}{D+1}} \ll D^{-1}. \quad (23)$$

We characterize this constraint by the dimensionless control parameter  $\xi$  defined by

$$\xi \equiv C_D^{-1}D(N\hbar/RE)^{\frac{D}{D+1}} \ll 1. \quad (24)$$

Solving Eqs. (19) and (24) for  $RE$ , one can express the system's area as

$$\frac{\mathcal{A}}{\hbar} = \frac{16\pi ND^{\frac{D+1}{D}}}{\eta(D-1)(\xi C_D)^{\frac{D+1}{D}}}. \quad (25)$$

Substituting (25) into (21), we realize that a violation of the holographic bound can occur if the number of spatial dimensions satisfies the inequality [20]

$$D \geq D^* \simeq 4\pi/\eta\xi^{1/D}. \quad (26)$$

Since the dimensionless control parameters satisfy the relations  $\xi \ll 1$  and  $\eta \ll 1$  [see Eqs. (19) and (24)], we learn from (26) that the critical dimension  $D^*$  (the minimal value of  $D$  above which a violation of the holographic bound can be realized) satisfies  $D^* \gg 4\pi$  [21].

Note that in the large  $D$  regime (26), the condition (21) for a violation of the holographic bound (1) can be simplified:

$$\frac{\mathcal{A}}{\hbar} < 32N\eta^D \sqrt{\frac{\pi}{2D}} \left( \frac{D}{4\pi} \right)^{D+1}. \quad (27)$$

Our analysis thus reveals that the holographic bound (1) can actually be violated in higher-dimensional space-times. But according to the arguments of Ref. [4], the holographic bound is necessary for the validity of the GSL, for otherwise the resulting black hole (after collapsing a shell upon the hyperentropic object) would have an

entropy  $S_{\text{BH}} = \mathcal{A}/4\hbar$  which is smaller than the entropy of the original (hyperentropic) system. We must therefore ask what was wrong with the original arguments of Susskind [4] suggesting that the holographic bound should follow from the GSL in any number of spatial dimensions. Can we resolve this apparent violation of the GSL?

We argue that one can escape a violation of the GSL if it turns out that it is actually not possible to form a stable (or meta-stable) black hole in the gedanken experiment of [4, 13]. Due to Hawking evaporation [8], a purely quantum effect, the black hole will have a *finite* lifetime. If it turns out that this lifetime is shorter than the relaxation time of the dynamically formed black hole, than a static (or quasi-static) black hole will never actually form in the gedanken experiment of [4, 13]. Instead, the intermediate non-equilibrium configuration will merely act as a catalyst for transforming the initial high entropy confined state into a final higher entropy state of unconfined Hawking radiation [13]. We shall now provide analytical estimates for the lifetime,  $\tau_{\text{bh}}$ , and dynamical relaxation time,  $\tau_{\text{relaxation}}$ , of  $(D+1)$ -dimensional black holes.

The Hawking radiation power emitted by a  $(D+1)$ -dimensional black hole of radius  $r_H$  can be approximated by the blackbody formula [22–24]

$$P_D = \sigma_D A_{\text{abs}} T^{D+1}, \quad (28)$$

where the  $D$ -dimensional Stefan-Boltzman constant is given by [22]

$$\sigma_D = \frac{A_{D-2}\Gamma(D+1)\zeta(D+1)N}{(2\pi)^D(D-1)\Gamma(\frac{D-1}{2})}. \quad (29)$$

Here

$$A_{\text{abs}} = \frac{A_{D-2}}{D-1} r_c^{D-1} \quad (30)$$

is the absorptive area of the black hole in the geometrical optics (high energy) limit [22, 23], where

$$r_c \equiv \left( \frac{D}{2} \right)^{\frac{1}{D-2}} \sqrt{\frac{D}{D-2}} r_H, \quad (31)$$

is the critical radius for null geodesics [22, 23] (if a photon travels inside this radius, it is captured by the black hole). It was recently shown [24] that the blackbody formula (28) provides a reasonably good description of the black-hole emission power. In fact, the agreement between the (numerically computed) black-hole power and the blackbody analytical formula (28) is very good in the large  $D$  regime (26) [24].

Substituting Eqs. (29)–(31) into Eq. (28) and using the relation [19]  $T_{\text{BH}} = \frac{(D-2)\hbar}{4\pi r_H}$  for the black hole's temperature, one finds

$$P_D \simeq N \left( \frac{D-2}{4\pi} \right)^{D+1} \left( \frac{r_c}{r_H} \right)^{D-1} \frac{D\zeta(D+1)\hbar}{\pi r_H^2} \quad (32)$$

for the total power radiated by a  $(D + 1)$ -dimensional black hole.

The corresponding decrease of the black hole mass during the Hawking evaporation is given by  $dM/dt = -P_D$ . Using the mass-radius relation (16) in Eq. (32), one may integrate this equation to find

$$\tau_{\text{bh}} \simeq \frac{2^{2D-2} \pi^{D+1} (D-1)}{D^2 (D-2)^D \zeta(D+1) \hbar N} \left( \frac{r_H}{r_c} \right)^{D-1} \mathcal{A} r_H \quad (33)$$

for the lifetime of the  $(D + 1)$ -dimensional black hole. Note that in the large  $D$  regime (26), one has  $(r_H/r_c)^{D-1} \simeq 2/De$ , which implies the compact expression

$$\tau_{\text{bh}} \simeq \left( \frac{4\pi}{D} \right)^{D+2} \frac{e}{32\pi \hbar N} \mathcal{A} r_H. \quad (34)$$

On the other hand, the characteristic timescale required for the dynamically formed black hole to settle down to a stationary, equilibrium configuration is given by [25–27]

$$\tau_{\text{relaxation}} = \Im \omega_0^{-1} = \frac{2\sqrt{D}}{D-2} \left( \frac{D}{2} \right)^{\frac{1}{D-2}} r_H, \quad (35)$$

where  $\omega_0$  is the fundamental black-hole quasinormal frequency [28]. Note that in the large  $D$  regime (26), one can approximate (35) by the compact formula

$$\tau_{\text{relaxation}} = \frac{2r_H}{\sqrt{D}}. \quad (36)$$

In order for a  $(D + 1)$ -dimensional black hole to be regarded as a stable (or meta-stable) state, its lifetime must be longer than its dynamical relaxation time:  $\tau_{\text{bh}} > \tau_{\text{relaxation}}$ . Thus, taking cognizance of Eqs. (34) and (36), one may deduce a lower bound on the area of stable (or meta-stable)  $(D + 1)$ -dimensional Schwarzschild black holes:

$$\left( \frac{\mathcal{A}}{\hbar} \right)_{\text{min}} \simeq \frac{16\sqrt{D}N}{e} \left( \frac{D}{4\pi} \right)^{D+1}. \quad (37)$$

As discussed above, one can accept a violation of the holographic entropy bound (the existence of hyperentropic physical systems) and at the same time avoid a

disturbing violation of the GSL in the gedanken experiment of [4, 13], provided the lifetime of the black hole which is formed from the collapse of the hyperentropic system is shorter than its relaxation time. In this case, a quasi-static black hole will never actually form in the gedanken experiment of [4, 13]. Instead, there would be an intermediate non-equilibrium configuration which will merely act as a catalyst for converting the initial high entropy confined state into a final higher entropy state of unconfined Hawking radiation [13]. Taking cognizance of Eq. (27), one realizes that the area of the black hole which would form from the collapse of the hyperentropic system is *smaller* than the minimal area (37) which is required for a meta-stable black-hole configuration. [The RHS of (37) is larger than the RHS of (27) by the factor  $\sim D/\eta^D$ .] Thus, we conclude that the GSL is *respected* despite the fact that the holographic bound (1) *can* be violated.

In summary, the gedanken experiment of [4] suggests that the holographic entropy bound (1) is necessary for the validity of the generalized second law of thermodynamics. However, in this work we have demonstrated explicitly that the bound (1) can be violated in higher-dimensional spacetimes. At first sight, this finding seems to open a possibility of violating the GSL in the gedanken experiment of [4]. However, our analysis reveals that hyperentropic systems are actually allowed to exist (they are harmless to the GSL) *provided* their area is bounded from above by:

$$\frac{\mathcal{A}}{\hbar} < \frac{16\sqrt{D}N}{e} \left( \frac{D}{4\pi} \right)^{D+1}. \quad (38)$$

It is of interest to search for other examples of physical systems which violate the holographic entropy bound (1). It would be highly important to verify that these systems do conform to the new area bound (38), which is necessary for the validity of the GSL.

## ACKNOWLEDGMENTS

This research is supported by the Meltzer Science Foundation. I thank Yael Oren and Arbel M. Ongo for helpful discussions. I thank Jacob D. Bekenstein for helpful correspondence.

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